2024

MATHEMATICS — GENERAL

Paper : DSE-B-1 and DSE-B-2

Candidates are required to give their answers in their own words as far as practicable.

ℕ *denotes the set of natural numbers.*

DSE-B-1

(Advanced Calculus)

Full Marks : 65

প্রান্তলিখিত সংখ্যাগুলি পূর্ণমান নির্দেশক।

১। সঠিক উত্তরটি লেখো ঃ ১×১০

(ক) যদি $\sum \frac{\sin nx}{n^p}$ শ্রেণিটি x-এর সকল বাস্তব মানের জন্য সমভাবে অভিসারী হয়, তখন $p =$

(অ) -1 (আ) $\frac{1}{2}$

(ই) $\frac{2}{3}$ (ঈ) 5

(খ) অপেক্ষকের অনুক্রমটি $\{f_n\}_n$, যেখানে $f_n(x) = x^n, 0 \le x \le 1$

(অ) $[0, 1]$ অন্তরালে সমভাবে অভিসারী নয়

(আ) $[0, 1]$ অন্তরালে সমভাবে অভিসারী

(ই) x-এর সকল বাস্তব মানের জন্য সমভাবে অভিসারী

(ঈ) উপরের কোনোটিই নয়।

(গ) $\sum nx^{n-1}$ ঘাত শ্রেণিটির অভিসরণ ব্যাসার্ধ হল

(অ) $\frac{1}{2}$ (আ) 1

(ই) $\frac{1}{4}$ (ঈ) 0

(ঘ) $f(x) = |\sin x|$ অপেক্ষকটির পর্যায়কাল হল

(অ) 2π (আ) $\pi/2$

(ই) 3π (ঈ) π

(ঙ) $x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots$ ঘাত শ্রেণিটি অভিসারী হবে যখন,

(অ) $-1<x<1$ (আ) $-1\le x<1$

(ই) $-1<x\le 1$ (ঈ) $-1\le x\le 1$

(চ) যদি $f(x)=x\sin x, -\pi\le x\le\pi$ অপেক্ষকটিকে Fourier শ্রেণিতে $\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$ হিসেবে প্রকাশ করা হয়, তখন a_0-এর মান হবে

(অ) 2 (আ) 0

(ই) 4 (ঈ) 1

(ছ) $L^{-1}\left\{\frac{4}{p-2}\right\}=$

(অ) $-4e^{2t}$ (আ) $4e^{-2t}$

(ই) $4e^{2t}$ (ঈ) $-4e^{-2t}$

(জ) যদি $L\{f(t)\}=\tan^{-1}\left(\frac{1}{p}\right)$, তখন $L\{tf(t)\}$ হল

(অ) $\tan^{-1}\left(\frac{1}{p^2}\right)$ (আ) $\frac{1}{1+p^2}$

(ই) $\frac{1}{1+p}$ (ঈ) $\tan^{-1}\left(\frac{2}{p\pi}\right)$

(ঝ) $L\{y''(t)\}$-এর মান হল

(অ) $\frac{d^2}{dp^2}L\{y\}$ (আ) $p^2L\{y\}-py(0)-y'(0)$

(ই) $p^2L\{y\}-py'(0)-y(0)$ (ঈ) $p^2L\{y\}-py(0)$

(ঞ) $\{f_n\}_n$ অপেক্ষকের অনুক্রমটির অভিসারী ডোমেন হয়, যেখানে $f_n(x)=\frac{x}{1+nx}, n\in\mathbb{N}$

(অ) $0<x<\infty$ (আ) $-\infty<x<0$

(ই) $-\infty<x<\infty$ (ঈ) $-1<x<1$

২। *যে-কোনো তিনটি* প্রশ্নের উত্তর দাও ঃ ৫×৩

(ক) $x+\frac{x^2}{2^2}+\frac{2!x^3}{3^3}+\frac{3!x^4}{4^4}+\ldots$ ঘাত শ্রেণিটির অভিসরণ ব্যাসার্ধ নির্ণয় করো।

(খ) দেখাও যে, $\left\{\frac{nx^2}{nx+1}\right\}$ অপেক্ষকের অনুক্রমটি [0, 1] অন্তরালে সমভাবে অভিসারী।

(গ) $f(x)=\begin{cases}0, & -\pi\le x<0\\ 1, & 0\le x\le\pi\end{cases}$ অপেক্ষকের Fourier শ্রেণিটি নির্ণয় করো।

(ঘ) $L\left\{\frac{2\left(e^t-\cos t\right)}{t}\right\}$-এর মান নির্ণয় করো।

(ঙ) $L^{-1}\left\{\log\left(\frac{s+a}{s+b}\right)\right\}$ নির্ণয় করো।

৩। *যে-কোনো চারটি* প্রশ্নের উত্তর দাও ঃ

(ক) (অ) দেখাও যে, $\{f_n\}_n$, $f_n(x)=\frac{nx}{1+n^2x^2}, 0\le x\le 1$, অপেক্ষকের অনুক্রমটি, যেখানে [0, 1] অন্তরালে বিন্দু অনুযায়ী অভিসারী হবে কিন্তু [0, 1] সমভাবে অন্তরালে অভিসারী হবে না।

(আ) অপেক্ষকের অনুক্রমের জন্য সমভাবে অভিসারী হওয়ার ক্ষেত্রে Cauchy-এর Condition-টি বিবৃত করো। $f_n(x)=\frac{n}{x+n}\ \forall x\in[0,1]$ হলে দেখাও যে, $\{f_n\}_n$ অপেক্ষকের অনুক্রমটি [0, 1] অন্তরালে সমভাবে অভিসারী হবে। ৫+(২+৩)

(খ) (অ) অপেক্ষক শ্রেণির সমভাবে অভিসারী হওয়ার জন্য Weierstrass' M-test-টি বিবৃত করো। দেখাও যে, $[0, a]$ অন্তরালে যে-কোনো $a>0$-এর জন্য $\sum_{n=1}^{\infty}\frac{nx^2}{n^3+x^3}$ শ্রেণিটি সমভাবে অভিসারী হবে।

(আ) দেখাও যে, $\log_e(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots$ এবং ঘাত শ্রেণিটির অভিসরণ ব্যাসার্ধ নির্ণয় করো।

(২+৩)+৫

(গ) (অ) ঘাত সংক্রান্ত Abel-এর উপপাদ্যটি Limit form-এ বিবৃত করো।

(আ) দেখাও যে, $\tan^{-1}x$-এর ঘাত শ্রেণিটি হল $\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots (-1 \le x \le 1)$। এর সাহায্যে দেখাও যে, $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$ ২+(৫+৩)

(ঘ) (অ) $-l \le x \le l$ অন্তরালে $f(x)$ অপেক্ষকটির জন্য Dirichlet's conditions বিবৃত করো।

(আ) $0 \le x \le \pi$ অন্তরালে $f(x) = \pi - x$ অপেক্ষকের Fourier cosine শ্রেণিটি নির্ণয় করো। এর সাহায্যে দেখাও যে, $1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots \text{to } \infty = \frac{\pi^2}{8}$। ২+(৫+৩)

(ঙ) (অ) যদি $L\{F(t)\} = f(p)$ হয়, তবে প্রমাণ করো যে, $L\{F(at)\} = \frac{1}{a} f\left(\frac{p}{a}\right)$। এর সাহায্যে $L\{\cos 6t\}$-এর মান নির্ণয় করো।

(আ) মান নির্ণয় করো ঃ $L^{-1}\left\{\frac{p}{(p-a)(p-b)}\right\} : a \ne b$ (৪+১)+৫

(চ) (অ) $L\left\{e^{-3t}\frac{\sin 2t}{t}\right\}$-এর মান নির্ণয় করো।

(আ) Laplace Transform-এর সাহায্যে নিম্নলিখিত অবকল সমীকরণটি সমাধান করো ঃ

$\frac{d^2y}{dt^2} + y = 8\cos t;\ y(0) = 1,\ y'(0) = -1$ ৪+৬

(ছ) (অ) $L^{-1}\left\{\frac{s^2}{\left(s^2 + a^2\right)^2}\right\}$-এর মান নির্ণয় করো।

(আ) Laplace Transform-এর সাহায্যে নিম্নলিখিত অবকল সমীকরণটি (ODE) সমাধান করো ঃ

$y'' - 3y' + 2y = 4e^{2t},\ y(0) = -3,\ y'(0) = 5$ ৪+৬

[English Version]

The figures in the margin indicate full marks.

1. Write the correct answer : 1×10

(a) If the series $\sum \frac{\sin nx}{n^p}$ is uniformly convergent for all real values of x, then $p =$

(i) -1 (ii) $\frac{1}{2}$

(iii) $\frac{2}{3}$ (iv) 5.

(b) The sequence $\{f_n\}_n$, where $f_n(x) = x^n, 0 \le x \le 1$ is

(i) not uniformly convergent on [0, 1]

(ii) uniformly convergent on [0, 1]

(iii) uniformly convergent for all real values of x

(iv) none of the above.

(c) The radius of convergence of the power series $\sum nx^{n-1}$ is

(i) $\frac{1}{2}$ (ii) 1

(iii) $\frac{1}{4}$ (iv) 0.

(d) The period of the function $f(x) = |\sin x|$ is

(i) 2π (ii) $\pi/2$

(iii) 3π (iv) π.

(e) The power series $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$ converges for

(i) $-1 < x < 1$ (ii) $-1 \le x < 1$

(iii) $-1 < x \le 1$ (iv) $-1 \le x \le 1$.

(f) If $f(x) = x\sin x, -\pi \le x \le \pi$ be presented in Fourier series as $\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$, then the value of a_0 will be

(i) 2 (ii) 0

(iii) 4 (iv) 1.

(g) $L^{-1}\left\{\frac{4}{p-2}\right\} =$

(i) $-4e^{2t}$ (ii) $4e^{-2t}$

(iii) $4e^{2t}$ (iv) $-4e^{-2t}$.

(h) If $L\{f(t)\} = \tan^{-1}\left(\frac{1}{p}\right)$, then $L\{tf(t)\}$ is

(i) $\tan^{-1}\left(\frac{1}{p^2}\right)$ (ii) $\frac{1}{1+p^2}$

(iii) $\frac{1}{1+p}$ (iv) $\tan^{-1}\left(\frac{2}{p\pi}\right)$.

(i) The value of $L\{y''(t)\}$ is

(i) $\frac{d^2}{dp^2}L\{y\}$ (ii) $p^2L\{y\} - py(0) - y'(0)$

(iii) $p^2L\{y\} - py'(0) - y(0)$ (iv) $p^2L\{y\} - py(0)$.

(j) The domain of convergence of the sequence of functions $\{f_n\}_n$, where $f_n(x) = \frac{x}{1+nx}, n \in \mathbb{N}$ is

(i) $0 < x < \infty$ (ii) $-\infty < x < 0$

(iii) $-\infty < x < \infty$ (iv) $-1 < x < 1$.

2. Answer ***any three*** questions : 5×3

(a) Find the radius of convergence of the series $x + \frac{x^2}{2^2} + \frac{2!x^3}{3^3} + \frac{3!x^4}{4^4} + \ldots$

(b) Show that the sequence of function $\left\{\frac{nx^2}{nx+1}\right\}$ on [0, 1] is uniformly convergent.

(c) Find the Fourier series for the function $f(x)=\begin{cases}0, & -\pi\le x<0\\ 1, & 0\le x\le\pi\end{cases}$.

(d) Find the Laplace transformation of $\left\{\frac{2\left(e^t-\cos t\right)}{t}\right\}$.

(e) Evaluate $L^{-1}\left\{\log\left(\frac{s+a}{s+b}\right)\right\}$.

3. Answer ***any four*** questions :

(a) (i) Show that the sequence of function $\{f_n\}_n$ defined by $f_n(x)=\frac{nx}{1+n^2x^2}, 0\le x\le 1$ is pointwise convergent on [0, 1] but not uniformly convergent on [0, 1].

(ii) State Cauchy's condition for uniform convergence of sequence of functions. If $f_n(x)=\frac{n}{x+n}\ \forall x\in[0,1]$, then show that the sequence of function $\{f_n\}_n$ converge uniformly on [0, 1]. 5+(2+3)

(b) (i) State Weierstrass' M-test for uniform convergence of series of functions. Show that the series $\sum_{n=1}^{\infty}\frac{nx^2}{n^3+x^3}$ is uniformly convergent on $[0, a]$ for any $a>0$.

(ii) Show that $\log_e(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots$ and find its radius of convergence. (2+3)+5

(c) (i) State Abel's theorem on Power Series in Limit form.

(ii) Show that the power series of $\tan^{-1}x$ is given by, $\tan^{-1}x=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\ldots(-1\le x\le 1)$.

Hence derive that $\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots$. 2+(5+3)

(d) (i) State the Dirichlet's conditions for a function $f(x)$ in an interval $-l \le x \le l$.

(ii) Find the Fourier cosine series of the function $f(x) = \pi - x$, $0 \le x \le \pi$ and hence deduce that

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots \text{ to } \infty = \frac{\pi^2}{8}.$$ 2+(5+3)

(e) (i) If $L\{F(t)\} = f(p)$, then prove that $L\{F(at)\} = \frac{1}{a} f\left(\frac{p}{a}\right)$, hence find $L\{\cos 6t\}$.

(ii) Find $L^{-1}\left\{\frac{p}{(p-a)(p-b)}\right\} : a \ne b$. (4+1)+5

(f) (i) Evaluate $L\left\{e^{-3t}\frac{\sin 2t}{t}\right\}$.

(ii) Using Laplace transform, solve the following differential equation :

$$\frac{d^2y}{dt^2} + y = 8\cos t;\ y(0) = 1,\ y'(0) = -1.$$ 4+6

(g) (i) Evaluate $L^{-1}\left\{\frac{s^2}{\left(s^2 + a^2\right)^2}\right\}$.

(ii) Using Laplace transform, solve the following ODE :

$$y'' - 3y' + 2y = 4e^{2t},\ y(0) = -3,\ y'(0) = 5.$$ 4+6

DSE-B-2

(Mathematical Finance)

Full Marks : 65

Group - A

(Marks : 10)

1. Choose the correct alternative : 1×10

(a) In Finance, risk is calculated by calculating the

(i) mean (ii) variance

(iii) standard deviation (iv) kurtosis.

(b) Regular interest payment to the bond holders is called

(i) Principal (ii) Coupon

(iii) Face value (iv) Yield.

(c) At what rate per cent per annum will a sum of money tripple in 16 years?

(i) 11% (ii) 11.5%

(iii) 12% (iv) 12.5%.

(d) The present value of ₹ 100 expected in two years from today at a discount rate of 5% is

(i) ₹ 105 (ii) ₹ 110.7

(iii) ₹ 95 (iv) ₹ 90.7.

(e) Suppose you invest ₹ 1000 in a savings account that pays 5% interest annually. How much will you have in the account after 3 years if the interest is compounded annually?

(i) 1150 (ii) 1157.62

(iii) 1152.50 (iv) 1200.

(f) Consider two linear equations $y = 2x + 3$ and $y = -2x + 5$. What is the correlation coefficient between the slope of these two lines?

(i) 1 (ii) 0

(iii) -1 (iv) Cannot be determined.

(g) The normalised version of the covariance is called

(i) Regression (ii) Correlation

(iii) Cross-section (iv) Spread.

(h) The ratio between the amount of profit and investment is called

(i) NPV (ii) Opportunity cost

(iii) Risk premium (iv) Rate of return.

(i) The average beta of all stocks in a market is

(i) -1 (ii) 0

(iii) 1 (iv) 1.5.

(j) The negative cash flows are classified as

(i) Present cash (ii) Future cash

(iii) Cash inflows (iv) Cash outflows.

Group - B

(Marks : 15)

Answer ***any three*** questions.

2. Write short notes on : (a) Bond Price (b) Short Selling. 2½+2½

3. Find the minimum value of the bivariate function $f(x, y) = x^2 + y^2$, subject to linear constraints $x + y = 1$ and $x - y = 1$ by using Lagrange's multiplier method. 5

4. Define Bond yield and Par yield. Suppose that 6-month, 12-month, 18-month and 24-month zero rates are 5%, 6%, 6.5% and 7% respectively. What is 2-year par yield? 1+1+3

(Given that $e^x = 0.9375, 0.9418, 0.9071, 0.8694$ for $x = -0.025, -0.06, -0.0975, -0.14$ respectiverly)

5. State and prove Arbitrage theorem. 1+4

6. Imagine that you are planning to retire in 35 years and you think you can afford to save ₹ 500 per month. Further, you believe that you can reasonably earn about 8% per year without taking too much risk. How much amount will you have accumulated at the time you retire? 5

Group - C

(Marks : 40)

Answer ***any four*** questions.

7. (a) What is a portfolio diagram? Derive the expression for portfolio mean return and variance.

(b) Consider a portfolio comprising of three securities in the following proportions and with the indicated security beta.

Security	Amount invested	Beta	Expected return
A	1.5 lakhs	1.0	12%
B	1.0 lakhs	1.5	13.5%
C	2 lakhs	0.8	9%

(i) What is the portfolio beta?

(ii) What is portfolio expected return? 2+2+6

8. Suppose Mr. A is considering an investment opportunity that requires an initial outlay of ₹ 10,000 and is expected to generate the following cash flow over five years :

Year 1 : ₹ 3,000

Year 2 : ₹ 4,000

Year 3 : ₹ 4,500

Year 4 : ₹ 5,000

Year 5 : ₹ 6,000

Using the Newton-Raphson method, calculate the Internal Rate of Return (IRR) for this investment opportunity. Provide your solution with a clear explanation of your calculations, including the initial guess for the IRR and the steps involved in each iteration of the Newton-Raphson method. 10

9. What do you mean by Conditional Value-at-Risk or CVAR? If the gain G from an investment is a random variable with mean m and standard deviation σ, then calculate the CVAR. 2+8

10. (a) What does duration tell you about the sensitivity of a bond portfolio to interest rates?

(b) A major lottery company advertise that it pays the winner ₹ 1 crore. The prize money is at the rate of ₹ 5 lakh each year (with the first payment is immediate) for a total of twenty payments. What is the present value of the prize at 10% interest rate compounding annually? 3+7

11. (a) If a \$ 5,000 at 5.5% coupon rate is to be purchased at a quoted price of \$ 4,350 to be redeemed in 14 years. What is the yield rate?

(b) State and prove annuity formula. 5+5

12. The correlation ρ between assets A and B is 0.1, and other data are given in the table below where $\rho = \frac{\sigma_{AB}}{\sigma_A \sigma_B}$, σ = standard deviation.

Asset	$\bar{r}$	σ
A	10%	15%
B	18%	30%

(a) Find the proportions α of A and $(1-\alpha)$ of B that define a portfolio of A and B having minimum standard deviation.

(b) What is the expected return of this portfolio?

(c) What is the value of this minimum standard deviation? 3+4+3

13. (a) Why is the non-diversifiable risk only relevant risk? How is such risk measured?

(b) The following facts are available :

$r_m = 0.14 \quad r_f = 0.0825 \quad r = 0.18.$

Compute the Beta coefficient (β). (2+2)+6